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Special symmetries of *jm* factors and *j* symbols

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Abstract. It has been suggested that the existence of one-dimensional irreps of a group leads to symmetries in the Racah algebra of the group. The familiar $2jm$ symbol arises as a special case of these symmetries where the one-dimensional irrep is the identity irrep. We derive the general result and give examples for the symmetric groups and for the point groups. These examples show that these new symmetries are more complicated than the previous suggestions imply.

1. Introduction

In the early 1960's both Hamermesh (1962) and Griffith (1962) noted that the existence of more than one one-dimensional irrep of a group will lead to symmetries in the Racah algebra of the group. More recently, van Zanten and de Vries (1970) mentioned these symmetries in connection with the class structure of finite groups, and Lulek and Lulek (1976a, b) have confirmed Griffith's work on the symmetries for the octahedral double group and its subgroups. In this paper we relate these special symmetries of the j and jm symbols to the permutation symmetries of the j and jm symbols as developed by Wigner (1940), Derome and Sharp (1965), Derome (1966), Butler (1975) and Butler and Wybourne (1976a, b).

The symmetries we obtain are more general than those of Hamermesh (1962) and Schindler and Mirman (1976). Further, their symmetries, derived only for the symmetric groups, do not hold simultaneously with permutation symmetries.

Many of the pure rotation point groups have several one-dimensional irreps and thus exhibit the symmetries discussed here. These symmetries could be used to reduce the size of the tables recently prepared (Butler 1979). However, the largest of the point groups is the icosahedral group and it has only one one-dimensional irrep, the identity, so the saving in space would not be great.

The case of the symmetric groups is rather more interesting. The symmetric groups are much used in nuclear calculations (see, for example, Vanagas 1971). They all have two one-dimensional irreps, the identity (scalar) irrep, $[n]$, and the alternating (antisymmetric) irrep, $[1^n]$.

In § 2 we discuss the properties of one-dimensional irreps in preparation for the definition of the Λ symbols in § 3. Sections 4 and 5 use these Λ symbols to derive symmetry relations for $3jm$ and $6j$ symbols. Section 6 discusses the symmetric groups as an example. Using the Young–Yamanouchi basis we derive the $\Lambda_{\lambda'}^{\tilde{\lambda}}$ phases for the $3jm$ factors for the imbedding $S_n \supset S_{n-1}$, thus factorising Hamermesh's result (Hamermesh 1962, p 266).

An essential requirement for Schur's lemma II (Hamermesh 1962, p 100) is that the bases of irrep spaces are chosen so that for similar irrep spaces the irrep matrices will be identical. For this reason we can label an irrep space with two labels, the label λ (the same for equivalent irreps) and a label (or set of labels)x, the same only if spaces are identical. Thus we have (Butler 1975, equation (2.4))

$$\mathrm{O}_R|x\lambda_i\rangle = \sum_j |x\lambda_j\rangle\lambda(R)_{ji}. \tag{1.1}$$

The product $V_{x_1\lambda_1}\otimes V_{x_2\lambda_2}$ of two irrep spaces is a representation space but a unitary transformation is required to reduce the space into a direct sum of spaces $\Sigma_{r\lambda} V_{(x_1\lambda_1x_2\lambda_2)r\lambda}$. Because of condition (1.1) the transformation can and is chosen to be independent of the x labels (Butler 1975, equation (3.2)):

$$|x_1\lambda_1l_1|x_2\lambda_2l_2\rangle = \sum_{r\lambda l} |(x_1\lambda_1, x_2\lambda_2)r\lambda l|\lambda_1l_1\lambda_2l_2\rangle. \tag{1.2}$$

The index r is required to distinguish sets of equivalent irrep spaces when they arise (Derome and Sharp 1965). Because the coupling coefficients are independent of the x_1 and x_2 labels, the separation and phases of the equivalent irreps $r\lambda$ as r varies must be fixed once and only once for each $\lambda_1\lambda_2$. This property can be used (Derome 1966) to deduce the permutation symmetries for the Racah–Wigner algebra of a group.

It is well known that jm symbols are easier to tabulate and use than are the coupling coefficients. The $2jm$ symbol

$$(\lambda)_{ll^*} = |\lambda|^{1/2}\langle 0|\lambda l, \lambda^*l^*\rangle, \tag{1.3}$$

where $|\lambda|$ is the dimension of the irrep λ, relates the phase properties of the complex conjugate irreps λ and λ^*. In particular, we have (Derome 1966, Butler 1975)

$$\lambda^*(R)_{ll'} = \sum_{mm'} (\lambda)^*_{ml}\lambda(R)^*_{mm'}(\lambda)_{m'l'}. \tag{1.4}$$

Note that $\lambda(R)^*_{mm'}$ is the complex conjugate of the matrix element $\lambda(R)_{mm'}$ whereas $\lambda^*(R)_{ll'}$ is the standard matrix element of some irrep λ^*. The $2jm$ symbols can always be chosen real (Butler and Wybourne 1976a) and, by suitable choice of $|\lambda^*l^*\rangle$ in terms of $|\lambda l\rangle^*$, of the form $(\lambda)_{ll'} = (\lambda)_{ll^*}\delta_{l'l^*}$.

The $3jm$ symbol defined by

$$\begin{pmatrix}\lambda_1 & \lambda_2 & \lambda\\ l_1 & l_2 & l\end{pmatrix}^r = |\lambda|^{-1/2}(\lambda^*)_{l^*l}\langle r\lambda^*l^*|\lambda_1l_1, \lambda_2l_2\rangle H(\lambda_1\lambda_2\lambda_3r) \tag{1.5}$$

cannot always be chosen real (Butler 1975). The phases and multiplicity separations of the $3jm$ symbols are chosen so the symbols have the highest possible symmetry properties. For SO_3 in the $SO_2(JM)$ basis, the coupling coefficients were computed before the recognition (Wigner 1940) of the $3jm$ symmetries, hence the presence of an historical phase, $H(J_1J_2J)_{SO_3} = (-)^{J_1-J_2+J}$.

The $6j$ and $9j$ symbols relate different schemes of coupling three or four irreps to give a fourth or fifth respectively. The relation of these $6j$ and $9j$ symbols to the appropriate overlap of the resultant bases, their relation with sums of products of $3jm$

symbols and the symmetry properties of j and jm symbols can all be found in Butler (1975).

2. One-dimensional irreps

This section discusses the multiplication properties of one-dimensional irreps. For every such irrep we can find a set of special $3jm$ symbols with properties similar to those of $2jm$ symbols.

Let ϵ be a one-dimensional irrep of a group and λ any irrep of the group. We name the representation obtained by the Kronecker product of λ^* on ϵ^* the tilde of λ:

$$\tilde{\lambda} = \lambda^* \times \epsilon^*. \tag{2.1}$$

With this definition $(\lambda\epsilon\tilde{\lambda})$ form a triad in the sense that $\lambda \times \epsilon \times \tilde{\lambda}$ contains the identity representation.

$\tilde{\lambda}$ is clearly a representation; to show that it is irreducible we use the following lemma.

Lemma. If Γ is a reducible representation, say $\Gamma = \gamma_1 + \gamma_2$ where γ_1 and γ_2 are representations, and μ is also a representation, then $\Gamma \times \mu$ is reducible.

Proof. By the distributive property of the Kronecker product

$$\Gamma \times \mu = (\gamma_1 + \gamma_2) \times \mu = (\gamma_1 \times \mu) + (\gamma_2 \times \mu), \tag{2.2}$$

but each of $\gamma_1 \times \mu$ and $\gamma_2 \times \mu$ are representations.

Theorem. If λ is irreducible then so is $\tilde{\lambda}$.

Proof. Using the associative laws after multiplication of $\tilde{\lambda}$ by ϵ

$$\tilde{\lambda} \times \epsilon = (\lambda^* \times \epsilon^*) \times \epsilon = \lambda^* \times (\epsilon^* \times \epsilon). \tag{2.3}$$

But $\epsilon \times \epsilon^*$ is one-dimensional and contains the identity irrep 0, so must be the identity irrep. Hence

$$\tilde{\lambda} \times \epsilon^* = \lambda^* \times \epsilon^* \times \epsilon = \lambda^* \times 0 = \lambda^*. \tag{2.4}$$

But by the lemma, if $\tilde{\lambda}$ is reducible so must λ^* (and λ) be reducible, which is a contradiction.

There are two possibilities for ϵ to consider, ϵ real and ϵ complex, giving

$$\epsilon^2 = 0 \qquad \text{or} \qquad \epsilon \times \epsilon^* = 0 \tag{2.5}$$

respectively. In both cases we have that $\tilde{\tilde{\lambda}} = \lambda$, for

$$\tilde{\tilde{\lambda}} = (\tilde{\lambda})^* \times \epsilon^* = (\lambda^* \times \epsilon^*)^* \times \epsilon^* = \lambda \times \epsilon \times \epsilon^* = \lambda, \tag{2.6}$$

but only for the real case do we have $(\widetilde{\lambda^*}) = (\tilde{\lambda})^*$, for

$$(\widetilde{\lambda^*}) = \lambda \times \epsilon^* \tag{2.7}$$

but

$$(\tilde{\lambda})^* = (\lambda^* \times \epsilon^*)^* = \lambda \times \epsilon. \tag{2.8}$$

Consider, for example, C_5, the cyclic group on five objects with its five one-dimensional irreps. With a particular choice of irrep ϵ we have the following character table, ($\omega = \exp \frac{2}{5}\pi i$):

C_5	1	C_5^1	C_5^2	C_5^3	C_5^4
0	1	1	1	1	1
ϵ	1	ω	ω^2	ω^3	ω^4
ϵ^*	1	ω^4	ω^3	ω^2	ω
λ	1	ω^3	ω	ω^4	ω^2
λ^*	1	ω^2	ω^4	ω	ω^3

The various products are easily evaluated to show that

$$\widetilde{(\lambda^*)} = \lambda \times \epsilon^* = \lambda^*$$

but

$$(\tilde{\lambda})^* = \lambda \times \epsilon = \epsilon^*.$$

In the following we denote the single partner of the irrep space V_ϵ as $|\epsilon\rangle$ and a typical partner of V_λ as $|\lambda l\rangle$. The coupling of V_ϵ and V_λ to $V_{(\epsilon\lambda)\tilde{\lambda}^*}$ may be performed as

$$|\lambda l\rangle|\epsilon\rangle = \sum_{l'} |(\lambda\epsilon)\tilde{\lambda}l'\rangle\langle\tilde{\lambda}^* l'|\lambda l, \epsilon\rangle. \tag{2.9}$$

The coupling coefficients $\langle\tilde{\lambda}^* l'|\lambda l, \epsilon\rangle$ are elements of the transformation from the product basis $|\lambda l\rangle\,|\epsilon\rangle$ to the standard basis for irrep $\tilde{\lambda}^*$. If this latter basis has no other conditions imposed upon it, we could certainly choose all $\langle\tilde{\lambda}^* l'|\lambda l, \epsilon\rangle$ as $+1$. This is not the case. $V_{(\lambda\epsilon)\tilde{\lambda}^*}$ may be equivalent to V_λ so the latter basis may depend directly on the former. In other instances a relationship may be indirect, for example by being a consequence of the complex conjugation symmetries. Thus we must explore whether the various requirements can be satisfied simultaneously.

If *abc* is a permutation π of 123 then we have (Butler 1975, equation (6.2))

$$\begin{pmatrix} \lambda_a & \lambda_b & \lambda_c \\ l_a & l_b & l_c \end{pmatrix}^r = \sum_s \{\pi\lambda_1\lambda_2\lambda_2\}_{rs} \begin{pmatrix} \lambda_1 & \lambda_2 & \lambda_3 \\ l_1 & l_2 & l_3 \end{pmatrix}^s. \tag{2.10}$$

Furthermore once chooses bases such that the various kets are partners of various irreps in a group–subgroup chain. If the group–subgroup branching has no multiplicity then it will force a one-to-one correspondence between $|\lambda l\rangle$ and $|\tilde{\lambda} l'\rangle = |\tilde{\lambda}\tilde{l}\rangle$. With a group–subgroup chain $G \supset H$ one writes a partner of irrep λ of G, σ of H as $|\lambda a\sigma i\rangle$. The ket $|\epsilon\rangle$ is now labelled $|\epsilon\eta\rangle$, η a label of a one-dimensional irrep of H. It follows that $\tilde{\sigma}^*$, defined as $\sigma \times \eta$, is an irrep of H. The coupling coefficients for the group G in the $G \supset H$ basis may be factorised into isoscalar factors for $G \supset H$ and coupling coefficients for H. The *jm* symbols factor in the same way into a *G–H–jm* factor and a *H–jm* symbol. For the remainder of this paper we write our results in terms of *jm* symbols to exploit their simpler symmetries. We make the choice of phase of Butler (1975, equation (5.2)), namely omitting the phase $H(\lambda_1\lambda_2\lambda_3 r)$ of (1.5).

3. Special 3*jm* factors, Λ symbols

We can now define a special $3jm$ symbol, the Λ symbol, as a generalisation of the $2jm$ symbol:

$$\Lambda^{\lambda}(\epsilon)_{ij}=|\lambda|^{1/2}\begin{pmatrix}\lambda & \tilde{\lambda} & \epsilon\\ i & j & 0\end{pmatrix}. \tag{3.1}$$

When ϵ is the identity irrep, 0, the Λ symbol is a $2jm$ symbol

$$\Lambda^{\lambda}(0)_{ij}=(\lambda)_{ij} \tag{3.2}$$

and $\tilde{\lambda}=\lambda^*$. By the factorisation mentioned above we have (Butler 1975, equation (13.2))

$$\begin{pmatrix}\lambda & \tilde{\lambda} & \epsilon\\ a & a' & \\ \sigma & \tilde{\sigma} & \eta\\ i & j & 0\end{pmatrix}^{G}_{H}=\begin{pmatrix}\lambda & \tilde{\lambda} & \epsilon\\ a & a' & \\ \sigma & \tilde{\sigma} & \eta\end{pmatrix}^{G}_{H}\begin{pmatrix}\sigma & \tilde{\sigma} & \eta\\ i & j & 0\end{pmatrix}^{H}. \tag{3.3}$$

We now define a Λ factor

$$\Lambda^{\lambda}(\epsilon\eta)_{a\sigma a'\tilde{\sigma}}=\frac{|\lambda|^{1/2}}{|\sigma|^{1/2}}\begin{pmatrix}\lambda & \tilde{\lambda} & \epsilon\\ a\sigma & a'\tilde{\sigma} & \eta\end{pmatrix}. \tag{3.4}$$

In cases where no ambiguity arises over a choice of ϵ we shall abbreviate $\Lambda^{\lambda}(\epsilon\eta)_{a\sigma a'\tilde{\sigma}}$ to $\Lambda^{\lambda}_{a\sigma a'\tilde{\sigma}}$; thus

$$\Lambda^{\lambda}_{a\sigma i a'\tilde{\sigma}j}=\Lambda^{\lambda}_{a\sigma a'\tilde{\sigma}}\Lambda^{\sigma}_{ij}. \tag{3.5}$$

From (2.9) and (3.1) we see the $\Lambda^{\lambda}_{a\sigma a'\tilde{\sigma}}$ are block-diagonal, each block being equal in dimension to the range of the branching multiplicity label a. We might expect that we can choose our basis kets such that the Λ factors will be diagonal in the branching multiplicity and real, because the $2jm$ factors are (Butler and Wybourne 1976a)

$$\Lambda^{\lambda}(0)_{a\sigma a'\sigma^*}=\pm\delta_{aa'}. \tag{3.6}$$

However, calculations show that while this is possible for $0L\supset T$, it is only true if care is taken in choosing the sign in (3.6).

4. Symmetries of 3*jm* factors

In this and the following sections we shall consider the case of ϵ real. Although our results are easily generalised to complex ϵ, or to several ϵ_i, the number of labels required complicates the notation.

The result of coupling λ_1^* and λ_2^* to some λ_3 may be compared with the coupling of $\tilde{\lambda}_1$ and $\tilde{\lambda}_2$ to λ_3. The precise relationship between the $3jm$ factors follows immediately from Butler (1975, equation (13.14)) or Butler and Wybourne (1976a, equation (41)):

$$
\begin{pmatrix} \tilde{\lambda}_1 & \tilde{\lambda}_2 & \lambda_3 \\ a_1' & a_2' & a_3 \\ \tilde{\sigma}_1 & \tilde{\sigma}_2 & \sigma_3 \end{pmatrix}^{r'}_{s'}
= \sum_{a_1 a_2 r s} |\lambda_1|^{1/2} |\lambda_2|^{1/2} |\sigma_1|^{1/2} |\sigma_2|^{1/2} \begin{Bmatrix} \tilde{\lambda}_1 & \tilde{\lambda}_2 & \lambda_3 \\ \lambda_2 & \lambda_1 & \epsilon \end{Bmatrix}_{00rr'}
$$

$$
\times (\lambda_1^*)_{a_1\sigma_1^* a_1 \sigma_1} (\lambda_2)_{a_2\sigma_2 a_2 \sigma_2^*} \Lambda^{\tilde{\lambda}_1}_{a_1'\tilde{\sigma}_1 a_1 \sigma_1} \Lambda^{\lambda_2}_{a_2\sigma_2 a_2' \tilde{\sigma}_2}
$$

$$
\times \begin{pmatrix} \lambda_2^* & \lambda_1^* & \lambda_3 \\ a_2 & a_1 & a_3 \\ \sigma_1^* & \sigma_2^* & \sigma_3 \end{pmatrix}^{r}_{s} \begin{Bmatrix} \tilde{\sigma}_1 & \tilde{\sigma}_2 & \sigma_3 \\ \sigma_2 & \sigma_1^* & \eta \end{Bmatrix}_{00ss'} . \tag{4.1}
$$

de Vries and van Zanten (1972) obtain a formula for the norm of $6j$ symbols of the form appearing here. However, their result, which is obtained from character theory, is valid only for multiplicity-free $6j$'s. Their result also follows immediately from the orthogonality relations of $6j$ symbols (Butler 1975, equation (9.11)). In such cases the dimension factors in (4.1) combine with the $6j$ norm to give the result that a $3jm$ with two tildes equals a product of phase factors and a $3jm$ with two stars.

Equation (4.1) modifies the symmetry relation derived by Hamermesh (1962, equation (7.216)) and Schindler and Mirman (1976a, equation (VIII.33)) for the symmetric groups. Their symmetry relation does not hold simultaneously with permutation symmetries as may be seen for the simple example of $S_3 \supset S_2$. The product in S_3, $[21]\times[21]$, contains $[3]$ in the symmetric part and $[1^3]$ in the antisymmetric part:

$$\{[21][21][3]\} = 1 \tag{4.2}$$

$$\{[21][21][1^3]\} = -1. \tag{4.3}$$

In S_2 we have that

$$\{[2][2][2]\} = \{[1^2][1^2][2]\} = 1 \tag{4.4}$$

so that the $3jm$ factor

$$\begin{pmatrix} [21] & [21] & [3] \\ [2] & [2] & [2] \end{pmatrix}$$

is unchanged on interchanging columns, but the $3jm$ factor

$$\begin{pmatrix} [21] & [21] & [1^3] \\ [2] & [1^2] & [1^2] \end{pmatrix}$$

changes sign. Noting $|\widetilde{[21]}\tilde{[2]}\rangle = |[21][1^2]\rangle$ and $|\tilde{[3]}\tilde{[2]}\rangle = |[1^3][1^2]\rangle$ for $\epsilon = [1^3]$ shows that the tilde symmetry cannot commute with the interchange symmetry, but note that both symmetries hold simultaneously.

5. Symmetries of $6j$ symbols

A similar symmetry relation to the above $3jm$ relation may be derived as a special case of the Biedenharn–Elliott sum rule (Butler and Wybourne 1976a, equation (27)):

$$
\begin{aligned}
\begin{Bmatrix} \lambda_1 & \lambda_2 & \lambda_3 \\ \tilde{\mu}_1 & \tilde{\mu}_2 & \tilde{\mu}_3 \end{Bmatrix}_{r_1 r_2 r_3 r_4}
&= \sum_{s_1 s_2 s_3 s_1' s_2' r_1' r_2'} |\mu_1|\,|\mu_2|\,|\mu_3| \phi_{\lambda_1} \phi_{\tilde{\mu}_1} \\
&\quad \times \{\mu_1 \tilde{\mu}_1 \epsilon\}\{\mu_2 \tilde{\mu}_2 \epsilon\}\{\mu_3 \tilde{\mu}_3 \epsilon\}\{(13)\lambda_1 \mu_2 \mu_3^*\} s_1 s_1' \{(23)\mu_1^* \lambda_2 \mu_3\} s_2 s_2' \\
&\quad \times \{\mu_1 \mu_2^* \lambda_3 s_3\}\{(123)\lambda_1 \tilde{\mu}_2^* \tilde{\mu}_3\} r_1 r_1' \{(132)\tilde{\mu}_1 \lambda_2 \tilde{\mu}_3^*\} r_2 r_2' \\
&\quad \times \begin{Bmatrix} \mu_2 & \tilde{\mu}_3^* & \lambda_1^* \\ \mu_3^* & \mu_2^* & \epsilon \end{Bmatrix}_{00 s_1' r_1'}
\begin{Bmatrix} \tilde{\mu}_3 & \tilde{\mu}_1^* & \lambda_2^* \\ \mu_1^* & \mu_3^* & \epsilon \end{Bmatrix}_{00 s_2' r_2'}
\begin{Bmatrix} \tilde{\mu}_1 & \tilde{\mu}_2^* & \lambda_3^* \\ \mu_2^* & \mu_1^* & \epsilon \end{Bmatrix}_{00 s_3' r_3} \\
&\quad \times \begin{Bmatrix} \lambda_1 & \lambda_2 & \lambda_3 \\ \mu_1^* & \mu_2^* & \mu_3^* \end{Bmatrix}_{s_1 s_2 s_3 r_4}.
\end{aligned}
\tag{5.1}
$$

This relationship between a $6j$ symbol and one with the bottom entries tilded is hardly simple enough to be called a symmetry relation. If the summations over the multiplicity indices could be shown to vanish then it would be of much use, but this is not so.

For example, the symmetric groups S_l for $l \geq 6$ are not simple phase so the sums over s_1' and s_2' survive for certain irreps (Butler 1975, equation (6)). Nevertheless, this equation is of some use in deciding upon multiplicity separations when permutation symmetries do not suffice.

Consider the calculation in Butler (1979) of the $6j$ symbols of the octahedral double group. The product $U' \times U'$ contains the irreps T_1 and T_2 twice each. The T_2 multiplicity is resolved by permutational symmetry, but the T_1 multiplicity is not. A first calculation of the table of $6j$ symbols used the resolution of the multiplicity obtained by requiring that the $6j$

$$
\begin{Bmatrix} T & U' & U' \\ T_1 & E' & E' \end{Bmatrix}_{0000}
$$

be zero. This led to values of the $6j$ symbols describing the tilde symmetry as if

$$
X_{rs} = \begin{Bmatrix} T_1 & U_1 & U_1 \\ U_1 & T_2 & A_2 \end{Bmatrix}_{00rs}; \tag{5.2}
$$

then

$$
X = \frac{1}{2\sqrt{15}} \begin{pmatrix} 2 & 1 \\ -1 & 2 \end{pmatrix}. \tag{5.3}
$$

It is clear that altering the multiplicity separation by taking a linear combination of the form

$$
X'_{rs'} = \sum_s X_{rs} U_{ss'} \tag{5.4}
$$

with

$$
U = \frac{1}{\sqrt{5}} \begin{pmatrix} 2 & 1 \\ 1 & -2 \end{pmatrix} \tag{5.5}
$$

will give a diagonal X'. This transformation, when applied to the entire set of $6j$

symbols, makes some $6j$ symbols zero which were not before, and some non-zero which were before. In particular, this new separation is equivalent to requiring that

$$\begin{Bmatrix} T_1 & U' & U' \\ E & E' & E' \end{Bmatrix}_{0001}$$

be zero. The new separation happens to give simpler numbers for the values of octahedral $6j$ symbols (only factors of 2 and 3 appear). The structure of the products of S_5 shows that a similar result holds for S_5, but for S_6 certain products related by the tilde operation have their own symmetry separations. No proof is available to show the two 'symmetries' give equivalent separations. To go back to the more general situation of § 3, where ϵ is not assumed real, the tetrahedral group provides a counter-example: $T\times T$ contains T twice, once each in the symmetric and the antisymmetric parts, and

$$\begin{Bmatrix} T & T & T \\ T & T & \rho_1 \end{Bmatrix}_{00rs}$$

is not diagonal in r, s.

6. The symmetric groups

In this section we use the Young–Yamanouchi basis to derive the values of the phases

$$\Lambda^{\lambda}([1^l],[1^{l-1}])_{\mu\tilde{\mu}} = \Lambda^{\lambda}_{\mu} \tag{6.1}$$

for the imbedding $S_l \supset S_{l-1}$. An account of the Young–Yamanouchi basis can be found in Hamermesh (1962, chapter 7). Partitions $(\lambda) = (\lambda_1, \lambda_2, \ldots, \lambda_j)$ of the integer l are used to label the irreps of S_l and a set of partitions $(\lambda)(\mu)\ldots$ will thus label a basis ket in the basis $S_l \supset S_{l-1} \supset \ldots$. The Y symbol is an abbreviated notation in which the basis ket is denoted by a series of integers listing where the cells are removed from a Young diagram in the reduction $S_n \supset S_{n-1}$ for $l \geqslant n \geqslant 1$. The two basis kets of the [21] irrep of S_3 are labelled

$$|[21][2][1]\rangle = |211\rangle$$

and

$$|[21][11][1]\rangle = |121\rangle.$$

Hamermesh obtains the value of the Λ phase for S_l in the Young–Yamanouchi basis as

$$\Lambda^{\lambda}(1^l)_{ij} = \Lambda^{\lambda}_i = (-)^{n_i}$$

where n_i is the number of interchanges of neighbouring elements required to change the Y symbol i into the Y symbol in natural order. Now this is the sum of the numbers of cells of lesser number below each cell. In view of this, the value of the Λ factor for $S_l \supset S_{l-1}$ is $\pm$ as the number of cells below the cell numbered l is even or odd.

The result may be compared with that of Butler and King (1973), who showed that if $\lambda = \tilde{\lambda}$ then λ^2 contains $[1^n]$ in the symmetric or antisymmetric part depending on whether there is an even or odd number of cells below the leading diagonal of the Young diagram.

7. Conclusion

In this paper we have shown how a one-dimensional irrep other than the identity irrep leads to symmetries in the Racah algebra of the group. We have extended the symmetries of Hamermesh for non-simply reducible groups and shown how they can be made to hold simultaneously with permutation symmetries. We have given examples which show that the application of the symmetries is not trivial, contrary to the speculations of Schlindler and Mirman (1976a, b). We have factorised Hamermesh's result for the Λ phases of the symmetric groups.

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